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New formulation of the classical dynamics of the relativistic string with massive ends

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Abstract. Dynamic equations in the theory of a relativistic string with point masses at the ends are formulated only in terms of geometric invariants of the world trajectories of the massive ends of the string (curvature k_a and torsion κ_a of the trajectories). These characteristics allow us to reproduce the string world surface up to its position in Minkowski space E_2^1. The torsions $\kappa_a(\tau)$, $a = 1,2$ obey a system of second-order differential equations with delay describing the retardation effects of the interaction of masses through the string, the k_a being constants. A new particular solution to these equations is found that corresponds to periodic torsions.

1. Introduction

A dynamic basis for the hadron model is the relativistic string with massive ends (for a review, see [1]). Until now no general solutions have been derived for the equations describing the dynamics of this system, therefore it seems of interest to consider a new mathematical formulation of that problem which would promote the investigation of its dynamics and the derivation of new exact solutions.

The action functional for a relativistic string with point masses at the ends results in equations of motion of the string and in boundary conditions that physically represent the equations of motion of two masses interacting through the string. An analogy arises between this system and classical electrodynamics with charges in which the field is described by Maxwell equations with charges and the dynamics of the charges interacting with the field is given by the Lorentz equations. Wheeler and Feynman [2], considering the action to propagate at a distance with a finite velocity, have eliminated the field variables from the equations of motion in electrodynamics, and have formulated the interaction between charges in terms of retarded and advanced propagation functions when there is no absorption and emission of the electromagnetic field.

For a system of a relativistic string with masses at the ends we may also utilize the principle of action at a distance to enable us to find equations of motion in terms of the characteristics of the trajectories along which the masses are moving provided the string variables are eliminated. It is clear that, as the problem is relativistic, it cannot be formulated within the equal-time formalism. In the simplest non-relativistic limit we arrive at a system of two masses coupled by a linearly rising potential [3].

In this paper, the classical dynamics of the relativistic string with massive ends in the d-dimensional Minkowski spacetime E_{d-1}^1 is reformulated in terms of the geometric invariants of both the string world sheet and the world lines of the point masses at the string ends. In the $d = 3$ case that we will examine here in more detail for simplicity, the string coordinates $x^\mu(\tau, \sigma)$ as functions of parameters τ and σ are completely defined by the constant curvatures k_a connected with the masses and the string tension and the torsions $\kappa_a(\tau)$, $a = 1, 2$ of the endpoint trajectories which are subjected to a system of second-order delay differential equations that takes account of the retardation effects of the interaction of two point masses through the string. The well known example [4–6] of the straight-line string with massive ends rotating in a given plane corresponds to a particular solution of this system with the constant torsions $\kappa_a(\tau) = \kappa_{0a}$, $a = 1, 2$ when the string ends are moving along the helices. In this case the string world sheet is a helicoid [6] in the three-dimensional spacetime E^1_2.

In addition, a new exact solution is also found for the periodic torsions $\kappa_a(\tau + 2\pi) = \kappa_a(\tau)$, $a = 1, 2$ which are given by the Weierstrass function with a real period proportional to 2π and a pure imaginary period $2\omega'$. The string coordinates are expressed in terms of normal elliptic integrals and describe a more intricate motion than the rotation of a stretched string in a given plane including its transverse vibrations. Just such motions ought to be considered in the string model of hadrons for the calculation of the contributions to the linear behaviour of the static interquark potential at long distances [7].

In section 2 the geometric approach to the classical dynamics of the relativistic string with massive ends is formulated in the Minkowski space E^1_{d-1} for any spacetime dimension d. Section 3 is devoted to the derivation of equations for trajectories of the massive string endpoints in the three-dimensional spacetime E^1_2. In section 4 the exact solution of these equations is obtained in the case of periodic torsions and the corresponding string world surface will be constructed in section 5. Section 6 contains some conclusions.

2. Equations of motion and boundary conditions

Consider the dynamics of a relativistic string with point masses m_1 and m_2 at the ends. The world sheet with coordinates $x^\mu(u^i)$, $\mu = 0, 1, \ldots, d-1$, $i = 0, 1$ swept out by the string in the d-dimensional Minkowski spacetime E^1_{d-1} is an extremal of the functional of the action [1, 6]

$$S = -\gamma \int_{\tau_1}^{\tau_2} \int_{\sigma_1(\tau)}^{\sigma_2(\tau)} \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2}\, \mathrm{d}\tau\, \mathrm{d}\sigma + \sum_{a=1}^{2} m_a \int_{\tau_1}^{\tau_2} \sqrt{\left(\frac{\mathrm{d}x^\mu(\tau, \sigma_a(\tau))}{\mathrm{d}\tau}\right)^2}\, \mathrm{d}\tau \tag{2.1}$$

where the first term is the action of a massless relativistic string, γ is the string tension, $u^i = (\tau, \sigma)$ are parameters on the string world surface, and the derivatives are as follows

$$\dot{x}^\mu = \frac{\partial x^\mu(\tau, \sigma)}{\partial \tau} \qquad x'^\mu = \frac{\partial x^\mu(\tau, \sigma)}{\partial \sigma}$$

$$\frac{\mathrm{d}x^\mu(\tau, \sigma_a(\tau))}{\mathrm{d}\tau} = \dot{x}^\mu(\tau, \sigma_a(\tau)) + \dot{\sigma}_a(\tau) x'^\mu(\tau, \sigma_a(\tau)) \qquad a = 1, 2.$$

The motion of the string endpoints in the plane of the parameters τ and σ is described by the functions $\sigma_a(\tau)$, $a = 1, 2$. As for a massless string, the action (2.1) is invariant under non-degenerate changes of variables, $\tilde{\tau} = \tilde{\tau}(\tau, \sigma)$ and $\tilde{\sigma} = \tilde{\sigma}(\tau, \sigma)$, which allows us to eliminate any two of the three independent components of the world sheet metric

$$g_{ij} = \eta_{\mu\nu} \frac{\partial x^\mu}{\partial u^i} \frac{\partial x^\nu}{\partial u^j} \qquad i, j = 1, 2. \tag{2.2}$$

It is convenient to introduce isothermal coordinates τ and σ in terms of which the metric (2.2) is diagonal and traceless

$$g_{00} + g_{11} = 0 \qquad g_{01} = g_{10} = 0. \tag{2.3}$$

The flat Minkowski metric $\eta_{\mu\nu}$ of the enveloping d-dimensional spacetime E^1_{d-1} has the signature $(+, -, \ldots, -)$.

Variation of the action (2.1) with respect to $x^\mu(\tau, \sigma)$ gives equations of motion linear in the gauge (2.3)

$$\ddot{x}^\mu(\tau, \sigma) - x''^\mu(\tau, \sigma) = 0 \tag{2.4}$$

and nonlinear boundary conditions at the string ends

$$\begin{aligned} m_1 \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\frac{\dot{x}^\mu(\tau, \sigma_1) + \dot{\sigma}_1(\tau) \acute{x}^\mu(\tau, \sigma_1)}{\sqrt{(1 - \dot{\sigma}_1^2(\tau))\, \dot{x}^2(\tau, \sigma_1)}} \right] &= \gamma[\acute{x}^\mu(\tau, \sigma_1) + \dot{\sigma}_1(\tau) \dot{x}^\mu(\tau, \sigma_1)] \\ m_2 \frac{\mathrm{d}}{\mathrm{d}\tau} \left[\frac{\dot{x}^\mu(\tau, \sigma_2) + \dot{\sigma}_2(\tau) \acute{x}^\mu(\tau, \sigma_2)}{\sqrt{(1 - \dot{\sigma}_2^2(\tau))\, \dot{x}^2(\tau, \sigma_2)}} \right] &= -\gamma[\acute{x}^\mu(\tau, \sigma_2) + \dot{\sigma}_2(\tau) \dot{x}^\mu(\tau, \sigma_2)]. \end{aligned} \tag{2.5}$$

Varying (2.1) with respect to $\sigma_a(\tau)$ we arrive at the same equations (2.5), therefore the functions $\sigma_a(\tau)$, $a = 1, 2$ are not dynamical variables [8].

A general solution to the equations of motion (2.4) and gauge conditions (2.3) is of the form

$$x^\mu(\tau, \sigma) = \tfrac{1}{2}[\psi^\mu_+(u^+) + \psi^\mu_-(u^-)] \qquad u^+ = \tau + \sigma, \; u^- = \tau - \sigma \tag{2.6}$$

where $\psi'^\mu_\pm(u^\pm)$ are two isotropic vectors,

$$\psi'^2_\pm(u^\pm) = 0 \tag{2.7}$$

tangent to the string world surface $x^\mu(\tau, \sigma)$. The conditions (2.7) may be satisfied if we represent $\psi'^\mu_\pm$ by the following expansions

$$\begin{aligned} \psi'^\mu_+(u^+) &= \frac{A_+(u^+)}{\sqrt{\sum_{\alpha=2}^{d-1} f'^2_\alpha(u^+)}} \left[e^\mu_0 + \tfrac{1}{2} e^\mu_1 \sum_{\alpha=2}^{d-1} f^2_\alpha(u^+) + \sum_{\alpha=2}^{d-1} e^\mu_\alpha f_\alpha(u^+) \right] \\ \psi'^\mu_-(u^-) &= \frac{A_-(u^-)}{\sqrt{\sum_{\alpha=2}^{d-1} g'^2_\alpha(u^-)}} \left[e^\mu_0 + \tfrac{1}{2} e^\mu_1 \sum_{\alpha=2}^{d-1} g^2_\alpha(u^-) + \sum_{\alpha=2}^{d-1} e^\mu_\alpha g_\alpha(u^-) \right] \end{aligned} \tag{2.8}$$

where the constant basis $\{e_0^\mu, e_1^\mu, e_\alpha^\mu\}$ is formed from two isotropic vectors $e_0^\mu, e_1^\mu, e_0^2 = 0, e_1^2 = 0, (e_0e_1) = 1$ and $(d-1)$ space-like vectors $e_\alpha^\mu, (e_\alpha e_\beta) = -\delta_{\alpha\beta}, (e_0e_\alpha) = (e_1e_\alpha) = 0, \alpha = 2, 3, \ldots, d-1$. The representations (2.8) fully define the world surface of a relativistic string without boundary in a d-dimensional Minkowski spacetime E_{d-1}^1 and allow us to construct its basic quadratic forms.

The expression for metric tensor (2.3) can be obtained by inserting (2.8) into $g_{00} = \dot{x}^2(\tau, \sigma) = \frac{1}{2}(\psi'_+(u^+)\psi'_-(u^-))$. In the three-dimensional spacetime E_2^1 with $d = 1+2$ and $f_2(u^+) = f(u^+)$, $g_2(u^-) = g(u^-)$, for example, the latter looks as follows

$$g_{00} = \dot{x}^2(\tau, \sigma) = \frac{A_+(u^+)A_-(u^-)}{4f'(u^+)g'(u^-)}[f(u^+) - g(u^-)]^2. \tag{2.9}$$

As is known [1], in the $d = 3$ case the Gauss equation for the world surface of the relativistic string $x^\mu(\tau, \sigma)$ reduces to the Liouville equation for $g_{00} = \dot{x}^2(\tau, \sigma)$

$$\frac{\partial^2 \ln \dot{x}^2(u^+, u^-)}{\partial u^+ \partial u^-} = \frac{A_+(u^+)A_-(u^-)}{2\dot{x}^2(u^+, u^-)} \tag{2.10}$$

and (2.9) is the general solution to this equation.

Computation of the coefficients of the second fundamental form

$$b_{\alpha|ij} = \left(n_\alpha \frac{\partial^2 x}{\partial u^i \partial u^j}\right) \qquad i, j = 0, 1;\ \alpha = 2, 3, \ldots, d-1 \tag{2.11}$$

requires a special choice of the orthonormalized system of $(d-2)$ unit normals n_α^μ

$$\left(n_\alpha \frac{\partial x}{\partial u^i}\right) = 0 \qquad (n_\alpha n_\beta) = -\delta_{\alpha\beta} \tag{2.12}$$

which together with tangent vectors $\dot{x}^\mu$ and $\acute{x}^\mu$ constitute a moving frame of reference. This can most easily be done for the $d-1 = 2$ case when the field of normals (2.12) contains only one vector $n^\mu(u^+, u^-)$ that may be constructed in terms of the vectors $\dot{x}^\mu$ and $\acute{x}^\mu$ as follows

$$n^\mu(u^+, u^-) = \frac{[\dot{x} \times \acute{x}]}{\dot{x}^2(u^+, u^-)} \tag{2.13}$$

where $[\dot{x} \times \acute{x}t] = \varepsilon_{\mu\nu\rho}\dot{x}^\nu \acute{x}^\rho$, and $\varepsilon^{\mu\nu\rho}$ is a totally antisymmetric unit tensor. Inserting the relations (2.8) with $d = 3$ into (2.13) and taking into account that $[e_0 \times e_1] = e_2$, $[e_1 \times e_2] = -e_1$, $[e_0 \times e_2] = e_0$ we arrive at the expansion of the normal n^μ over the isotropic basis $\{e_0^\mu, e_1^\mu, e_2^\mu\}$

$$n^\mu(u^+, u^-) = \frac{2e_0^\mu + f(u^+)g(u^-)e_1^\mu + [f(u^+) + g(u^-)]\, e_2^\mu}{f(u^+) - g(u^-)}. \tag{2.14}$$

Using the expansions (2.8) with $d = 3$ and (2.14) for coefficients of the second quadratic form $b_{2|ij} = b_{ij}$ of the string world surface $x^\mu(\tau, \sigma)$, according to (2.11) we obtain

$$b_{00} = b_{11} = \frac{A_+(u^+) - A_-(u^-)}{2} \qquad b_{01} = b_{10} = \frac{A_+(u^+) + A_-(u^-)}{2}. \tag{2.15}$$

The first equality of (2.15) shows that the string world surface belongs to the class of the minimal surfaces [9] because its mean curvature is zero

$$h = \frac{1}{2} b_{ij} g^{ij} = \frac{b_{00} - b_{11}}{2 g_{00}} = 0. \tag{2.16}$$

Here it is assumed that for any point of the string world sheet there holds the condition $g_{00} = \dot{x}^2 > 0$ or $f'(u^+) g'(u^-) > 0$ and $f(u^+) \neq g(u^-)$ as follows from (2.9).

For an arbitrary dimensionality d of the enveloping spacetime E^1_{d-1} the condition of minimality (2.16) in the coordinate system (2.3) should be replaced by the relations

$$h_\alpha = \frac{b_{\alpha|00} - b_{\alpha|11}}{2 g_{00}} = 0 \qquad \alpha = 2, 3, \ldots, d-1. \tag{2.17}$$

For a relativistic string with massive ends the coordinates $x^\mu(\tau, \sigma)$ of the minimal string world surface obey the nonlinear boundary conditions (2.5). Substituting (2.6) into (2.5) for the isotropic vectors (2.7) and functions $\sigma_a(\tau)$ we get

$$(-1)^{a+1} m_a \frac{\mathrm{d}}{\mathrm{d}\tau} \left\{ \frac{\psi'_+(u_a^+) \dot{u}_a^+ + \psi'_-(u_a^-) \dot{u}_a^-}{\sqrt{\frac{1}{2} \psi'_+(u_a^+) \psi'_-(u_a^-) \dot{u}_a^+ \dot{u}_a^-}} \right\} = \gamma [\psi'^\mu_+(u_a^+) \dot{u}_a^+ - \psi'^\mu_-(u_a^-) \dot{u}_a^-] \tag{2.18}$$

$$u_a^+ = \tau + \sigma_a(\tau) \qquad u_a^- = \tau - \sigma_a(\tau) \qquad a = 1, 2.$$

For each of value $a = 1, 2$, only $d - 1$ of the d equations (2.18) are independent of each other since the projections of the system (2.18) onto the tangent vectors

$$\dot{x}^\mu(u^+, u^-) = \frac{\psi'^\mu_+(u^+) + \psi'^\mu_-(u^-)}{2} \qquad \acute{x}^\mu(u^+, u^-) = \frac{\psi'^\mu_+(u^+) - \psi'^\mu_-(u^-)}{2}$$

coincide. Thus, $2(d-1)$ independent equations of the system (2.18) contain, as $2d$ unknown quantities, two functions $\sigma_a(\tau)$ and $2(d-1)$ independent components of the isotropic vectors $\psi'^\mu_\pm$ expressed, according to (2.8), through $A_\pm, f_\alpha, g_\alpha$ which are, as we see from the boundary conditions (2.18), functions of $\sigma_a(\tau)$. That indefiniteness is a consequence of equations (2.7) and (2.18) under conformal transformations of the parameters $\tilde{u}^\pm = \tilde{u}^\pm(u^\pm)$ where $\tilde{u}^\pm(u^\pm)$ are two arbitrary functions of one variable. So, the definition of system (2.18) may be supplemented by imposing two auxiliary conditions which must depend on one variable. Choosing the equalities

$$A_+(u^+) = A_-(u^-) = A = \text{constant} \tag{2.19}$$

to be gauge conditions we will completely fix the coordinate system $u^i = (\tau, \sigma)$ on the string world sheet. In three dimensions, $d = 3$, this choice of gauge has a simple geometric meaning. Indeed, having fixed $A_\pm(u^\pm)$ according to (2.19), the coefficients of the second quadratic form $b_{2|ij} = b_{ij}$ at $d = 3$ (2.15) become

$$b_{00} = (n\ddot{x}) = 0 \qquad b_{01} = (n\dot{x}') = A. \tag{2.20}$$

Geometrically [9], the conditions (2.20) mean that the isothermal coordinates (2.3) are at the same time the asymptotic lines on the string world surface.

Let us show that by making the gauge setting (2.19) we can fix the functions $\sigma_a(\tau)$ in equations (2.18). In fact, projecting (2.18) onto normals n^μ_α, $\alpha = 2, 3, \ldots, d-1$ and taking account of (2.6) and (2.11) one finds $2(d-1)$ equations

$$(1+\dot\sigma_a^2(\tau))b_{\alpha|00}(\tau,\sigma_a(\tau)) + 2\dot\sigma_a(\tau)b_{\alpha|01}(\tau,\sigma_a(\tau)) = 0 \qquad \alpha = 2,3,\ldots,d-1 \qquad a = 1,2. \tag{2.21}$$

In the $d = 3$ case when $n^\mu_2 = n^\mu$, $b_{2|ij} = b_{ij}$ equations (2.21) with (2.20) reads

$$\dot\sigma_a(\tau) = 0 \qquad a = 1,2. \tag{2.22}$$

Consequently, the σ_a are constants and we put $\sigma_1 = 0$ and $\sigma_2 = \pi$. For $d \geq 4$ from (2.21) we may also derive equations (2.22) using the arbitrariness in choice of the field of normals n^μ_α corresponding to the group of transformations $\mathrm{SO}(d-2)$. In fact, utilizing the expansions (2.8) for the vectors $\psi'^\mu_\pm$ we get

$$(\ddot x \pm \dot x')^2 = \psi''^2_\pm(u^\pm) = -A^2_\pm(u^\pm)$$

which in the gauge (2.19) and in the metric (2.3) imply

$$x^2_{;00} + x^2_{;01} = -A^2 \qquad (x_{;00}x_{;01}) = 0 \tag{2.23}$$

where the semicolon stands for a covariant differentiation with respect to the metric (2.2). Therefore, when $d \geq 4$, we may, without loss of generality, direct the normals n^μ_2 and n^μ_3 along two mutually orthogonal space-like vectors $x^\mu_{;01}$ and $x^\mu_{;00}$ respectively: $n^\mu_2 \sim x^\mu_{;01}$, $n^\mu_3 \sim x^\mu_{;00}$. As a result, the coefficients of the second quadratic form (2.11) become equal

$$\begin{aligned} &b_{2|00} = 0 \qquad b_{2|01} = -\sqrt{-x^2_{;01}} \\ &b_{3|00} = -\sqrt{-x^2_{;00}} \qquad b_{3|01} = 0 \\ &b_{\alpha|ij} = 0 \qquad \alpha = 4,5,\ldots,d-1. \end{aligned} \tag{2.24}$$

With the latter equalities, equations (2.21) for $\alpha = 4, 5, \ldots, d-1$ are identically satisfied, and for $\alpha = 2, 3$ take the form

$$(1+\dot\sigma_a^2(\tau))x^2_{;00}(\tau,\sigma_a(\tau)) = 0 \qquad 4\dot\sigma_a^2(\tau)x^2_{;01}(\tau,\sigma_a(\tau)) = 0$$

whence, owing to (2.23), we obtain equations (2.22) and, setting $\sigma_a = (0, \pi)$, the conditions

$$x^2_{;00}(\tau, 0) = x^2_{;00}(\tau,\pi) = 0. \tag{2.25}$$

The $2(d-2)$ functions $f_\alpha(u^+)$ and $g_\alpha(u^-)$, $\alpha = 2, 3, \ldots, d-1$ remaining upon gauge (2.19) will obey two conditions (2.25) and $2(d-4)$ relations (2.24) when $d \geq 4$, and also two projections of the boundary conditions (2.18) on the tangent vectors $\dot x^\mu$

and $\acute{x}^\mu$. For projecting it is convenient to employ the conditions (2.5) that with the use of (2.22) may be written in the form

$$\ddot{x}^\mu(\tau,0) - \frac{(\dot{x}\ddot{x})}{\dot{x}^2}\dot{x}^\mu(\tau,0) = \frac{\gamma}{m_1}\sqrt{\dot{x}^2}\acute{x}^\mu(\tau,0) \qquad \sigma = 0$$

$$\ddot{x}^\mu(\tau,\pi) - \frac{(\dot{x}\ddot{x})}{\dot{x}^2}\dot{x}^\mu(\tau,\pi) = -\frac{\gamma}{m_2}\sqrt{\dot{x}^2}\acute{x}^\mu(\tau,\pi) \qquad \sigma = \pi. \tag{2.26}$$

Taking advantage of the conformal gauge (2.3) and equations of motion (2.4) it is easy to show that the projections (2.26) onto $\dot{x}^\mu(\tau,\sigma_a)$ vanish, and projections onto $\acute{x}^\mu(\tau,\sigma_a)$ give the equations

$$\frac{\partial}{\partial\sigma}\left(\frac{1}{\sqrt{\dot{x}^2(\tau,\sigma)}}\right)_{|\sigma=\sigma_a} = (-1)^a\frac{\gamma}{m_a} \qquad a = 1,2. \tag{2.27}$$

For the case of a three-dimensional Minkowski space the functions $f_2(u^+) = f(u^+)$ and $g_2(u^-) = g(u^-)$ in expansions (2.8) should obey two equations (2.27). For the $d = 4$ case equations (2.27) are to be supplemented with two conditions (2.25) for four unknown functions $f_\alpha(u^+), g_\alpha(u^-)$, $\alpha = 2,3$ from expansions (2.8). Finally, for the general d case in addition to (2.27) and (2.25) there are $2(d-4)$ equations (2.24) with $d \geq 4$. Thus, the dynamics of the relativistic string with massive ends in the Minkowski space E^1_{d-1} is described by the system of $2(d-2)$ equations (2.27), (2.25) and (2.24) with $d \geq 4$ being more complicated with growing dimensionality d of the spacetime E^1_{d-1}. Therefore, it is natural at first to examine the simplest equations from this list, equations (2.27), in the case of propagation of the relativistic string with massive ends in a three-dimensional Minkowski space E^1_2.

3. Equations for trajectories of a string with massive ends in a three-dimensional spacetime

Let us dwell upon the case of a three-dimensional spacetime E^1_2 when coordinates (2.6) of the minimal surface of a relativistic string with massive ends in the representation (2.8) and gauge (2.19) are defined by two functions $f(u^+)$ and $g(u^-)$ that obey the boundary conditions (2.27). Inserting the general solution (2.9) of the Liouville equation (2.10) into (2.27) we obtain the system of two delay differential equations for the functions $f(\tau)$ and $g(\tau)$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\frac{g'(\tau)}{f'(\tau)} + 2\frac{f'(\tau)+g'(\tau)}{f(\tau)-g(\tau)} = \frac{\gamma}{m_1}|A|\frac{|f(\tau)-g(\tau)|}{\sqrt{f'(\tau)g'(\tau)}} \tag{3.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\frac{g'(\tau-2\pi)}{f'(\tau)} + 2\frac{f'(\tau)+g'(\tau-2\pi)}{f(\tau)-g(\tau-2\pi)} = -\frac{\gamma}{m_2}|A|\frac{|f(\tau)-g(\tau-2\pi)|}{\sqrt{f'(\tau)g'(\tau-2\pi)}}. \tag{3.2}$$

For $m_1 = m_2 = 0$ the system of (3.1) and (3.2) has periodic solutions $g(\tau) = f(\tau)$, $f(\tau) = f(\tau + 2\pi)$ that according to (2.9) violate the minimality condition (2.16) at the points $\sigma = \sigma_a$, $a = 1,2$, and conversely, if one of the functions, either $f(\tau)$ or $g(\tau)$, is periodic, the other is also periodic and $m_1 = m_2 = 0$. Therefore, periodic solutions to equations (3.1) and (3.2) can exist only for a massless string [8].

The system (3.1) and (3.2) respects invariance under the same Möbius transformation of the functions $f(u^+)$ and $g(u^-)$

$$\bar{f}(u^+) = \frac{af(u^+) + b}{cf(u^+) + d} \qquad \bar{g}(u^-) = \frac{ag(u^-) + b}{cg(u^-) + d} \qquad ad - bc = 1 \tag{3.3}$$

which corresponds to the relativistic invariance of the underlying string theory. In fact, it is easy to see that the Lorentz transformations of vectors $\psi'^{\mu}_{\pm}$ and, according to (2.8), the vectors of the isotropic basis $\{e^\mu_0, e^\mu_1, e^\mu_2\}$ as well, induce the transformations (3.3) of the functions $f(u^+)$ and $g(u^-)$. Therefore relativistically invariant expressions, for instance (2.9), in terms of the functions $f(\tau)$ and $g(\tau)$ should be invariants with respect to (3.3).

Now let us demonstrate that the minimal surface of the string is fully defined by the world trajectories $x^\mu(\tau, \sigma_a)$ of its massive endpoints. To this ends, for $d = 3$ we shall describe the trajectories in terms of geometric invariants, curvature k_a and torsion κ_a. As is well known [9], these characteristics uniquely define a curve in a three-dimensional space up to its position. In the general the curvature of a curve $x^\mu(\tau)$ in the pseudo-Euclidean spacetime is given by the following expression [9]

$$k(\tau) = \frac{1}{\dot{x}^2}\sqrt{\frac{(\dot{x}\ddot{x})^2}{\dot{x}^2} - \ddot{x}^2} = \frac{1}{\dot{x}^2}\left\{-\left[\ddot{x}^\mu - \frac{(\dot{x}\ddot{x})}{\dot{x}^2}\dot{x}^\mu\right]^2\right\}^{1/2}.$$

Substituting the left-hand side of equations (2.26) for $x^\mu(\tau, \sigma_a)$, $a = 1, 2$ into this formula and using the conditions (2.3) we obtain

$$k_a = \frac{\gamma}{m_a} \qquad a = 1, 2. \tag{3.4}$$

Torsion of an arbitrary curve $x^\mu(\tau)$ in the pseudo-Euclidean spacetime is defined as [9]

$$\kappa(\tau) = \frac{\varepsilon_{\mu\nu\sigma}\dot{x}^\mu \ddot{x}^\nu \dot{\ddot{x}}^\sigma}{(\dot{x}\ddot{x})^2 - \dot{x}^2\ddot{x}^2}.$$

Differentiating equations (2.26) with respect to τ and inserting the expressions for $\ddot{x}^\mu(\tau, \sigma_a)$ and $\dot{\ddot{x}}^\mu(\tau, \sigma_a)$, $a = 1, 2$, the torsions of the trajectories $x^\mu(\tau, \sigma_a)$ can be written as

$$\kappa_a(\tau) = \frac{\varepsilon_{\mu\nu\sigma}\dot{x}^\mu \dot{x}'^\nu \dot{x}'^\sigma}{(\dot{x}^2(\tau, \sigma_a))^2} \qquad a = 1, 2$$

which, owing to the definitions (2.11) and (2.13) and condition (2.21), are reduced to the form

$$\kappa_a(\tau) = \frac{A}{\dot{x}^2(\tau, \sigma_a)} \qquad a = 1, 2. \tag{3.5}$$

Substituting $\dot{x}^2(\tau, \sigma_a)$ from (2.9) into (3.5) we obtain the expressions for torsions

$$\kappa_1(\tau) = \frac{4f'(\tau)g'(\tau)}{A[f(\tau) - g(\tau)]^2} \tag{3.6}$$

$$\kappa_2(\tau - \pi) = \frac{4f'(\tau)g'(\tau - 2\pi)}{A[f(\tau) - g(\tau - 2\pi)]^2} \tag{3.7}$$

invariant under the transformations (3.3).

Formulae (3.6) and (3.7) together with equations (3.1) and (3.2) allow us to express the functions $f(\tau)$ and $g(\tau)$ in terms of the torsions $\kappa_a(\tau)$ as follows. Calculating from (3.6) and (3.7) the differences of the functions

$$\frac{1}{|f(\tau)-g(\tau)|}=\frac{\sqrt{A\kappa_1(\tau)}}{2\sqrt{f'(\tau)g'(\tau)}}\qquad \frac{1}{|f(\tau)-g(\tau-2\pi)|}=\frac{\sqrt{A\kappa_2(\tau-\pi)}}{2\sqrt{f'(\tau)g'(\tau-2\pi)}} \tag{3.8}$$

and then inserting them into the boundary conditions (3.1) and (3.2) with allowance made for (3.4), we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\frac{g'(\tau)}{f'(\tau)}+\varepsilon_1\sqrt{A\kappa_1(\tau)}\left(\sqrt{\frac{f'(\tau)}{g'(\tau)}}+\sqrt{\frac{g'(\tau)}{f'(\tau)}}\right)=2k_1\sqrt{\frac{A}{\kappa_1}}\\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\frac{g'(\tau-2\pi)}{f'(\tau)}+\varepsilon_2\sqrt{A\kappa_2(\tau-\pi)}\left(\sqrt{\frac{f'(\tau)}{g'(\tau-2\pi)}}+\sqrt{\frac{g'(\tau-2\pi)}{f'(\tau)}}\right)\\
&\qquad=\ -2k_2\sqrt{\frac{A}{\kappa_2(\tau-\pi)}}
\end{aligned} \tag{3.9}$$

where ε_a, $a=1,2$ are the signs of the products $f'(\tau)[f(\tau)-g(\tau)]$ and $f'(\tau)[f(\tau)-g(\tau-2\pi)]$, respectively. Taking the logarithm and differentiating with respect to τ, formulae (3.6) and (3.7) with the use of (3.8) are transformed to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(f'(\tau)g'(\tau))-\varepsilon_1\sqrt{A\kappa_1(\tau)}\left(\sqrt{\frac{f'(\tau)}{g'(\tau)}}-\sqrt{\frac{g'(\tau)}{f'(\tau)}}\right)=\frac{\kappa_1'(\tau)}{\kappa_1(\tau)}\\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\ln(f'(\tau)g'(\tau-2\pi))-\varepsilon_2\sqrt{A\kappa_2(\tau-\pi)}\left(\sqrt{\frac{f'(\tau)}{g'(\tau-2\pi)}}-\sqrt{\frac{g'(\tau-2\pi)}{f'(\tau)}}\right)\\
&\qquad=\ \frac{\kappa_2'(\tau-\pi)}{\kappa_2(\tau-\pi)}.
\end{aligned} \tag{3.10}$$

The sum and difference of (3.9) and (3.10) give the following system of equations for the first boundary

$$\begin{aligned}
&\left[2\frac{\mathrm{d}}{\mathrm{d}\tau}-\sqrt{\kappa_1(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{\kappa_1(\tau)}}\right)-k_1\sqrt{\frac{A}{\kappa_1(\tau)}}\right]\frac{1}{\sqrt{f'(\tau)}}=-\varepsilon_1\sqrt{A\kappa_1(\tau)}\frac{1}{\sqrt{g'(\tau)}}\\
&\left[2\frac{\mathrm{d}}{\mathrm{d}\tau}-\sqrt{\kappa_1(\tau)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{\kappa_1(\tau)}}\right)+k_1\sqrt{\frac{A}{\kappa_1(\tau)}}\right]\frac{1}{\sqrt{g'(\tau)}}=\varepsilon_1\sqrt{A\kappa_1(\tau)}\frac{1}{\sqrt{f'(\tau)}}
\end{aligned} \tag{3.11}$$

and for the second boundary

$$\begin{aligned}
&\left[2\frac{\mathrm{d}}{\mathrm{d}\tau} - \sqrt{\kappa_2(\tau-\pi)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{\kappa_2(\tau-\pi)}}\right) + k_2\sqrt{\frac{A}{\kappa_2(\tau-\pi)}}\right]\frac{1}{\sqrt{f'(\tau)}} \\
&\qquad = -\varepsilon_2\sqrt{A\kappa_2(\tau-\pi)}\frac{1}{\sqrt{g'(\tau-2\pi)}} \\
&\left[2\frac{\mathrm{d}}{\mathrm{d}\tau} - \sqrt{\kappa_2(\tau-\pi)}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\sqrt{\kappa_2(\tau-\pi)}}\right) - k_2\sqrt{\frac{A}{\kappa_2(\tau-\pi)}}\right]\frac{1}{\sqrt{g'(\tau-2\pi)}} \\
&\qquad = \varepsilon_2\sqrt{A\kappa_2(\tau-\pi)}\frac{1}{\sqrt{f'(\tau)}}.
\end{aligned} \tag{3.12}$$

Finally, eliminating $1/\sqrt{f'(\tau)}$ and then $1/\sqrt{g'(\tau)}$ from (3.11) we arrive at the equations which connect $f(\tau)$ and $g(\tau)$ with the torsion $\kappa_1(\tau)$

$$\begin{aligned}
D(f(\tau)) &= D\left(\int^\tau \sqrt{A\kappa_1(\eta)}\,\mathrm{d}\eta\right) + \frac{\kappa_1(\tau)}{2}\left(1 - \frac{k_1^2}{\kappa_1^2(\tau)}\right) - 2k_1\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} \\
D(g(\tau)) &= D\left(\int^\tau \sqrt{A\kappa_1(\eta)}\,\mathrm{d}\eta\right) + \frac{\kappa_1(\tau)}{2}\left(1 - \frac{k_1^2}{\kappa_1^2(\tau)}\right) + 2k_1\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_1(\tau)}}.
\end{aligned} \tag{3.13}$$

The same procedure applied to equations (3.12) allows us to express $f(\tau)$ and $g(\tau-2\pi)$ in terms of $\kappa_2(\tau)$ as follows

$$\begin{aligned}
D(f(\tau)) &= D\left(\int^\tau \sqrt{A\kappa_2(\eta-\pi)}\,\mathrm{d}\eta\right) + \frac{\kappa_2(\tau-\pi)}{2} \\
&\qquad \times\left(1 - \frac{k_2^2}{\kappa_2^2(\tau-\pi)}\right) + 2k_2\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_2(\tau-\pi)}} \\
D(g(\tau-2\pi)) &= D\left(\int^\tau \sqrt{A\kappa_2(\eta-\pi)}\,\mathrm{d}\eta\right) + \frac{\kappa_2(\tau-\pi)}{2} \\
&\qquad \times\left(1 - \frac{k_2^2}{\kappa_2^2(\tau-\pi)}\right) - 2k_2\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_2(\tau-\pi)}}.
\end{aligned} \tag{3.14}$$

In formulae (3.13) and (3.14) we made use of the Schwarzian derivative

$$D(f(\tau)) = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2 = -2\sqrt{f'(\tau)}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{1}{\sqrt{f'(\tau)}}\right). \tag{3.15}$$

Thus, the functions $f(\tau)$ and $g(\tau)$ and therefore, according to (2.6) and (2.8), the string coordinates $x^\mu(\tau\sigma)$ are completely defined by the torsions $\kappa_a(\tau)$, $a = 1, 2$ of the world trajectories of massive string endpoints.

Eliminating $D(f)$ and $D(g)$ from the four equations (3.13) and (3.14) we obtain for the torsions $\kappa_a(\tau)$, $a = 1, 2$ the following two delay differential equations of second order

$$
\begin{aligned}
D(f(\tau)) &= D\left(\int^{\tau}\sqrt{A\kappa_1(\eta)}\,\mathrm{d}\eta\right) + \frac{\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right) - 2k_1\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} \\
&= D\left(\int^{\tau}\sqrt{A\kappa_2(\eta-\pi)}\,\mathrm{d}\eta\right) + \frac{\kappa_2(\tau-\pi)}{2} \\
&\quad\times\left(1-\frac{k_2^2}{\kappa_2^2(\tau-\pi)}\right) + 2k_2\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_2(\tau-\pi)}}
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
D(g(\tau)) &= D\left(\int^{\tau}\sqrt{A\kappa_1(\eta)}\,\mathrm{d}\eta\right) + \frac{\kappa_1(\tau)}{2}\left(1-\frac{k_1^2}{\kappa_1^2(\tau)}\right) + 2k_1\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_1(\tau)}} \\
&= D\left(\int^{\tau}\sqrt{A\kappa_2(\eta+\pi)}\,\mathrm{d}\eta\right) + \frac{\kappa_2(\tau+\pi)}{2} \\
&\quad\times\left(1-\frac{k_2^2}{\kappa_2^2(\tau+\pi)}\right) - 2k_2\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{\frac{A}{\kappa_2(\tau+\pi)}}.
\end{aligned}
\tag{3.17}
$$

Thus, in the framework of the geometrical method, the classical dynamics of the relativistic string with massive ends in the three-dimensional Minkowski space E_2^1 is described by two delay differential equations of second order (3.16) and (3.17). This system is of fundamental importance in searching the world surfaces of a relativistic string with massive ends in the ambient three-dimensional space E_2^1. Specifically, it follows from these equations that inside the interval $0 \leq \tau \leq \pi$ the torsions $\kappa_a(\tau)$ are arbitrary functions and in order to specify uniquely a solution of equations (3.16) and (3.17), they should be fixed there as the initial data by the choice of the initial position $x^\mu(0,\sigma)$ and initial velocity $\dot{x}^\mu(0,\sigma)$, $0 < \sigma < \pi$ of the string [6]. Continuation of these functions outside the interval $0 < \tau < \pi$ is made by the integrals of equations (3.16) and (3.17), so that two conditions of smoothness at the points 0 and π for the continued functions $\kappa_a(\tau)$, $-\infty < \tau < +\infty$, may always be fulfilled with the four arbitrary constants.

The simplest solution to equations (3.16) and (3.17) are constant torsions $\kappa_a(\tau) = \kappa_{0a}$ when the ends of the string are moving along helices obeying the following conditions [6]

$$
\kappa_{01}\left(1-\frac{k_1^2}{\kappa_{01}^2}\right) = \kappa_{02}\left(1-\frac{k_2^2}{\kappa_{02}^2}\right).
\tag{3.18}
$$

In this case we obtain from equations (3.13) and (3.14) the equalities

$$
D(g(\tau)) = D(f(\tau)) = D(g(\tau-2\pi))
\tag{3.19}
$$

whence, as explained in [10], it follows that the functions $f(\tau)$, $g(\tau)$ and $g(\tau-2\pi)$ are related by the Möbius transformations

$$
g(\tau) = \frac{\alpha_1 f(\tau)+\beta_1}{\gamma_1 f(\tau)+\delta_1} = \frac{\alpha_2 g(\tau-2\pi)+\beta_2}{\gamma_2 g(\tau-2\pi)+\delta_2}.
\tag{3.20}
$$

The constant coefficients in (3.20)—$\alpha_a, \beta_a, \gamma_a, \delta_a$—obey the normalization conditions $\alpha_a\delta_a - \beta_a\gamma_a = 1$ and two relations following from the boundary conditions (3.1) and (3.2). The world surface $x^\mu(\tau,\sigma)$ of the relativistic string with massive ends turns out to be a helicoid in the three-dimensional space E_2^1 [6].

4. Trajectories with periodic torsions

It turns out that the system of (3.16) and (3.17) possesses smooth periodic solutions

$$\kappa_a(\tau) = \kappa_a(\tau + 2\pi). \tag{4.1}$$

In fact, with the use of (4.1) we may write the sum and difference of (3.16) and (3.17) in the following form

$$D\left(\int^{\tau}\sqrt{A\kappa_1(\eta)}\,\mathrm{d}\eta\right) + \frac{\kappa_1(\tau)}{2}\left(1 - \frac{k_1^2}{\kappa_1^2(\tau)}\right) = D\left(\int^{\tau}\sqrt{A\kappa_2(\eta+\pi)}\,\mathrm{d}\eta\right) + \frac{\kappa_2(\tau+\pi)}{2}\left(1 - \frac{k_2^2}{\kappa_2^2(\tau+\pi)}\right) \tag{4.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(k_1\sqrt{\frac{A}{\kappa_1(\tau)}} + k_2\sqrt{\frac{A}{\kappa_2(\tau\pm\pi)}}\right) = 0. \tag{4.3}$$

From (4.3) one finds the integral of motion

$$\frac{k_1}{\sqrt{\kappa_1(\tau)}} + \frac{k_2}{\sqrt{\kappa_2(\tau\pm\pi)}} = k^2 \tag{4.4}$$

where k^2 is an arbitrary positive constant. Note that relations (4.4) contain only one arbitrary constant k^2 so that the smoothness of the curves $\kappa_a(\tau)$ continued outside the interval $0 < \tau < \pi$ cannot be guaranteed. In this case the equalities (4.1) and (4.4) may give rise to discontinuous solutions for $\kappa_a(\tau)$ over the whole real axis $-\infty < \tau < +\infty$, which are not considered here.

In the class of smooth functions we find for the torsions $\kappa_a(\tau)$, $a = 1, 2$, in the interval $0 < \tau < \pi$, the following representation

$$\begin{aligned} \frac{1}{\sqrt{\kappa_1(\tau)}} &= \frac{k^2}{k_1 + k_2|p(\tau)|} \\ \frac{1}{\sqrt{\kappa_2(\tau\pm\pi)}} &= \frac{k^2|p(\tau)|}{k_1 + k_2|p(\tau)|} \end{aligned} \tag{4.5}$$

which makes (4.4) an identity. The real-valued function $p(\tau)$ is defined by equations (4.1) and (4.2). Let us show that $p(\tau)$ is periodic $p(\tau) = p(\tau + 2\pi)$, and can be extended smoothly to the whole real axis τ. Inserting (4.5) into (4.2) we obtain the second-order differential equation for $p(\tau)$

$$p(\tau)p''(\tau) - \left[\frac{1}{2} + \frac{k_2|p(\tau)|}{k_1 + k_2|p(\tau)|}\right]p'^2(\tau) + \frac{1}{2}\left[\frac{(p^2(\tau)-1)(k_1 + k_2|p(\tau)|)^2}{k^4} - \frac{k^4p^2(\tau)(k_1 - k_2|p(\tau)|)}{(k_1 + k_2|p(\tau)|)}\right] = 0. \tag{4.6}$$

The substitution

$$p'^2(\tau) = \phi(p) \tag{4.7}$$

changes (4.6) to a first-order equation for the function $\phi(p)$, and integrating the latter over p we obtain

$$\phi(p) = k^4 p^2(\tau) - \frac{\Delta(p)}{k^4}(k_1 + k_2|p(\tau)|)^2 \boxminus w^2(p) \tag{4.8}$$

where

$$\Delta(p) = p^2(\tau) - 2\rho p(\tau) + 1 \tag{4.9}$$

and ρ is the integration constant. Now the function $p(\tau)$ is defined by (4.7), the right-hand side of which, (4.8), is a polynomial of the fourth degree in $p(\tau)$ with real-valued coefficients and positive $\phi(p) \equiv w^2(p) > 0$ for real $p(\tau)$. After putting $p(\tau) = 0$ (4.8) becomes $w^2(0) = -k_1^2/k^4 < 0$, whence it follows that $p(\tau)$ takes values either on the half-line $p(\tau) > 0$ or on $p(\tau) < 0$. The latter in turn ensures that the coefficients of polynomial (4.8) are fixed in sign.

As is known [12], the solution of equation (4.7) can be represented in terms of elliptic functions with periods 2ω and $2\omega'$. To this end, for simplicity we consider the case of equal masses at the string ends, $m_1 = m_2$ when, according to (3.4) $k_1 = k_2$, and one puts $k^4 = k_1|q|$, where q is an arbitrary constant, and $E = 2[1 + \varepsilon(p)\rho]$, $q^2/4 + E > 4$. In this case the elliptic curve (4.8) has two mutually inverse real-valued positive roots

$$p_1 = \frac{1+\sqrt{1-4\lambda}}{1-\sqrt{1-4\lambda}} \qquad p_2 = p_1^{-1} \tag{4.10}$$

where

$$\lambda = \frac{-E + \sqrt{E^2 + 4q^2}}{2q^2} \qquad 0 < \lambda < \frac{1}{4}.$$

With the use of (4.10) the solution to equation (4.7) may be represented as follows

$$|p(\tau)| = p_1 + \frac{[\phi'(|p|)/4]_{|p|=p_1}}{\{\wp(\tau) - [\phi''(|p|)/24]_{|p|=p_1}\}}. \tag{4.11}$$

Here $\wp(\tau) = \wp(\tau, g_2, g_3)$ is the Weierstrass function with real period 2ω and pure imaginary period $2\omega'$, g_2 and g_3 are real-valued invariants of the polynomial (4.8), $g_2^3 - 27g_3^2 > 0$. In the interval $0 < \tau < \omega$, when $e_1 < \wp(\tau) < \infty$, where

$$e_1 = \wp(\omega) = \frac{k_1}{12|q|}(q^2 + 2E) > 0 \tag{4.12}$$

by virtue of $e_1 > [\phi''(|p|)/24]_{|p|=p_1}$, the function (4.11) is smooth and monotonically decreasing from a maximum $|p(0)| = p_1$ at point $\tau = 0$ down to a minimum $|p(\omega)| = p_2 = p_1^{-1}$ at point $\tau = \omega$ and has at most three points of inflection. In accordance with the properties of the Weierstrass function [12], outside the interval $0 < \tau < \omega$ the function $|p(\tau)|$ continues with period 2ω in an even manner

$$|p(-\tau)| = |p(\tau)| \tag{4.13}$$

and to the whole real axis τ periodically with period 2ω

$$|p(\tau+2\omega)| = |p(\tau)|. \tag{4.14}$$

The lines p_1 and $p_2 = p_1^{-1}$ are envelopes of the curve (4.11).

Thus, formula (4.5) supplemented with (4.11), according to (4.13) and (4.14), defines the torsions $\kappa_a(\tau)$ as smooth 2ω-periodic even functions

$$\kappa_a(-\tau) = \kappa_a(\tau) \qquad \kappa_a(\tau+2\omega) = \kappa_a(\tau) \tag{4.15}$$

for all real values of the evolution parameter τ. To fulfil equalities (4.1), the real half-period ω of the function (4.11) is to be fixed at π that results in the following condition on the arbitrary constants ρ and q

$$\omega = \int_{e_1}^{\infty} \frac{\mathrm{d}\tau}{\sqrt{4t^3 - g_2 t - g_3}} = \pi. \tag{4.16}$$

The properties of torsions, (4.15) and (4.16), together with expressions (3.5) for the metric-tensor component of the string world surface (2.3) imply

$$\dot{x}^2(-\tau,\sigma_a) = \dot{x}^2(\tau,\sigma_a) \qquad \dot{x}^2(\tau+2\pi,\sigma_a) = \dot{x}^2(\tau,\sigma_a) \qquad a = 1,2. \tag{4.17}$$

To close this section, we note that, in the $m_1 = m_2$ case, the motion of the string ends proceeds along similar curves with $k_1 = k_2$ and $\kappa_1(\tau) = \kappa_2(\tau)$. In fact, the function (4.11) obeys a simple rule of addition

$$|p(\tau \pm \pi)| = \frac{1}{|p(\tau)|}. \tag{4.18}$$

Substitution of (4.18) into expression (4.5) for $1/\kappa_2(\tau)$ gives

$$\kappa_2(\tau) = \frac{k}{|q|}\,[1 + |p(\tau)|]^2 = \kappa_1(\tau) \tag{4.19}$$

whence with (3.5), it follows that

$$\dot{x}^2(\tau,0) = \dot{x}^2(\tau,\pi) \tag{4.20}$$

which is none other than the equality of lengths of trajectories of the masses in equal intervals of τ

$$l_1 = \int_{\tau_1}^{\tau_2} \sqrt{\dot{x}^2(\tau,0)}\,\mathrm{d}\tau = \int_{\tau_1}^{\tau_2} \sqrt{\dot{x}^2(\tau,\pi)}\,\mathrm{d}\tau = l_2.$$

5. Definition of the string world surface

We shall define the functions $f(\tau)$ and $g(\tau)$ from equations (3.13) and (3.14) taking into account that their right-hand sides are periodic owing to (4.1). Therefore the left-hand sides of these equations, i.e. the Schwarzian derivatives of the functions $f(\tau)$ and $g(\tau)$, are periodic as well, $D(f(\tau)) = D(f(\tau + 2\pi))$, $D(g(\tau)) = D(g(\tau + 2\pi))$, whence, as explained in [10], it follows that

$$f(\tau + 2\pi) = \frac{af(\tau) + b}{cf(\tau) + d} \qquad g(\tau + 2\pi) = \frac{ag(\tau) + b}{cg(\tau) + d} \qquad ad - bc = 1. \tag{5.1}$$

The coefficients of these Möbius transformations are taken to be the same so that the torsions (3.6) and (3.7) obey the condition (4.1). Specifying $b = c = 0$ and $a = d$, from (5.1) we obtain the periodic functions $f(\tau + 2\pi) = f(\tau)$ and $g(\tau + 2\pi) = g(\tau)$ corresponding to the case of the massless string.

In the general case, the real-valued pairs of solutions $(f(\tau), g(\tau))$ and $(f(\tau + 2\pi), g(\tau + 2\pi))$ for $(a + d) \geq 2$ have either one or two points of intersection given by the equation

$$F(f) = cf^2(\tau) + (d - a)f(\tau) - b = 0 \tag{5.2}$$

whereas for $(a + d) < 2$ they do not intersect at all. With (5.1) the expression (3.7) for $\kappa_2(\tau \pm \pi)$ assumes the form

$$\kappa_2(\tau \pm \pi) = \frac{4f'(\tau)g'(\tau)}{A[(af(\tau) + b) - g(\tau)(cf(\tau) + d)]^2}. \tag{5.3}$$

Expressing $4f'(\tau)g'(\tau)$ in terms of $\kappa_1(\tau)$ from formula (3.6) and inserting it back into (5.3) and using the notation $\kappa_1(\tau)/\kappa_2(\tau) = p^2(\tau)$ we arrive at the equation quadratic in $g(\tau)$

$$p^2(\tau)(f(\tau) - g(\tau))^2 = [(af(\tau) + b) - g(\tau)(cf(\tau) + d)]^2. \tag{5.4}$$

Two roots of this equation correspond to two different choices of the sign of function $p(\tau)$ and can be written as a common expression

$$g(\tau) = \frac{(a - p(\tau))f(\tau) + b}{cf(\tau) + (d - p(\tau))} \qquad ad - bc = 1 \tag{5.5}$$

whose coefficients, in contrast to the case of constant torsions (see formula (3.20)), depend on τ and form a matrix with the determinant

$$\Delta = p^2(\tau) - (a + d)p(\tau) + 1. \tag{5.6}$$

Comparing (5.1) with (5.5) we get the equality

$$g(\tau) = \frac{ag(\tau - 2\pi) + b}{cg(\tau - 2\pi) + d} = \frac{(a - p(\tau))f(\tau) + b}{cf(\tau) + (d - p(\tau))}$$

whence it follows that

$$g(\tau - 2\pi) = \frac{(p^{-1}(\tau) - d)f(\tau) + b}{cf(\tau) + (p^{-1}(\tau) - a)}. \tag{5.7}$$

Substituting (5.5) and (5.7) into formulae (3.6) and (3.7), respectively, and changing $\kappa_1(\tau)$ and $\kappa_2(\tau \pm \pi)$ by the expressions (4.5) we obtain the equation

$$\Delta(p)\left(\frac{f'(\tau)}{F(f)}\right)^2 - p'(\tau)\left(\frac{f'(\tau)}{F(f)}\right) - \frac{(k_1 + k_2|p(\tau)|)^2}{4k^4} = 0 \tag{5.8}$$

that defines the two-valued function $(f'(\tau)/F(f))$ in terms of $p(\tau)$ and $p'(\tau)$ as follows

$$\frac{f'(\tau)}{F(f)} = \frac{p'(\tau) \pm \sqrt{p'^2(\tau) + \Delta(p)(k_1 + k_2|p(\tau)|)^2/k^4}}{2\Delta(p)}. \tag{5.9}$$

Using (5.5), (5.7) and (5.9), it is easy to show that the boundary conditions (3.1) and (3.2) reproduce equations (4.7) and (4.8) with the constant ρ expressed in terms of the coefficients of transformation (5.1). Inserting (5.5) and (5.7) into equations (3.1) and (3.2) we represent their sum and difference in the form

$$\begin{aligned} 2\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\left[\Delta(p) - p'(\tau)\frac{F(f)}{f'(\tau)}\right] - 4(p(\tau) - p^{-1}(\tau))\frac{f'(\tau)}{F(f)} - 2\frac{p'(\tau)}{p(\tau)} \\ = \frac{(k_1 - k_2|p(\tau)|)}{\sqrt{\Delta(p)(f'(\tau)/F(f))^2 - p'(\tau)(f'(\tau)/F(f))}} \end{aligned} \tag{5.10}$$

$$\frac{2[p'(\tau) - 2\Delta(p)(f'(\tau)/F(f))]}{p(\tau)} = \frac{(k_1 + k_2|p(\tau)|)}{\sqrt{\Delta(p)(f'(\tau)/F(f))^2 - p'(\tau)(f'(\tau)/F(f))}}. \tag{5.11}$$

Substitution of (5.9) into (5.11) gives

$$\pm\sqrt{p'^2(\tau) + \frac{\Delta(p)}{k^4}(k_1 + k_2|p(\tau)|)^2} = -k^2 p(\tau) \tag{5.12}$$

where the sign of the root is determined by that of the function $p(\tau)$. After comparing (5.6) with (4.9) and identifying

$$2\rho = a + d \tag{5.13}$$

(5.12) is easily recognized as (4.7). Upon substitution of (5.9) into (5.10) the latter takes the form (4.6) and is also reduced to (4.7) and (4.8). Thus, the function $p(\tau)$ with (5.13) is defined by the representation (4.11).

Using (4.11) we now determine the functions $f(\tau)$ and $g(\tau)$. Owing to (5.12) the expression (5.9) assumes the form

$$\frac{f'(\tau)}{F(f)} = \frac{p'(\tau) - k^2 p(\tau)}{2\Delta(p)}. \tag{5.14}$$

To express the function $g(\tau)$ in terms of $p(\tau)$ we consider the relationship

$$\frac{g'(\tau)}{G(g)} = \frac{g'(\tau)}{cg^2(\tau) + (d-a)g(\tau) - b}. \tag{5.15}$$

Substitution of expression (5.5) and (5.14) into (5.15) gives

$$\frac{g'(\tau)}{G(g)} = -\frac{p'(\tau) + k^2 p(\tau)}{2\Delta(p)}. \tag{5.16}$$

Integrating (5.14) and (5.15) we get

$$\begin{aligned} \int^{f(\tau)} \frac{\mathrm{d}f}{F(f)} &= c_1 + \frac{1}{2}\int^{p(\tau)} \frac{\mathrm{d}p}{\Delta(p)} - \frac{k^2}{2}J(\tau). \\ \int^{g(\tau)} \frac{\mathrm{d}g}{G(g)} &= c_2 - \frac{1}{2}\int^{p(\tau)} \frac{\mathrm{d}p}{\Delta(p)} - \frac{k^2}{2}J(\tau). \end{aligned} \tag{5.17}$$

Here the integrals are depending on the conditions $|a+d| > 2$, $|a+d| = 2$ or $|a+d| < 2$ performed in terms of the same elementary functions since the discriminants of polynomials (5.2), (5.15) and (5.6) coincide, and the elliptic integral

$$J(\tau) = -\int_{p_1}^{|p(\tau)|} \frac{p\,\mathrm{d}p}{\Delta(p)w(p)} \tag{5.18}$$

with the use of (4.11), is split into a sum of normal elliptic integrals of the first and third kind. Solutions (5.17) should be periodic up to the Möbius transformations (5.1). The latter may, depending on whether $|a+d| > 2$, $|a+d| = 2$ or $|a+d| < 2$, always be reduced either to the hyperbolic, or parabolic, or elliptic form respectively by Möbius transformations (3.3) (see, e.g., [10]). Then inserting (5.17) into (5.1) with (4.13) leads to the constraint on arbitrary constants $a+d$ and q

$$\varepsilon(p)k^2\Omega(\frac{a+d}{2}, q) = R(a+d) \tag{5.19}$$

in addition to (4.16). Here Ω is a real period of the integral (5.18), and a function R depends on the choice of parametrization of the coefficients in (5.1). Finally, the coordinates $x^\mu(\tau,\sigma)$ of the minimal surface of the relativistic string with massive ends are given via expressions (5.17) for the functions $f(\tau)$ and $g(\tau)$ by formulae (2.8) and (2.9).

6. Conclusion

In this paper, it has been shown that the world sheet of a relativistic string with massive ends is completely defined by trajectories of its massive endpoints. In a three-dimensional Minkowski space E_2^1 these trajectories are characterized by two geometric invariants, a constant geodesic curvature and torsion that is generally a function of the evolution parameter τ on the string surface. When the torsions of these trajectories are constants, our approach allows us to reproduce a well-known exact solution describing the rotation of a straight-line string with massive ends in a given plane [4–6]. In this case the trajectories of motion of the massive endpoints turn out to be helices and the surface is a helicoid in E_2^1. It is worth mentioning that the helicoid is the only non-trivial minimal surface belonging to the class of ruled surfaces generated by the motion of a straight lines in a space. Therefore a solution of that sort does not describe transverse excitations of the string and hence does not contribute to the linear behaviour of the static potential between quarks at long distances.

The new solution we have found here describes a more intricate motion of the string when the massive endpoints moving along the same world trajectories with a constant curvatures and a periodic torsions. In this case the string world surface is not a helicoid and does not belong to the class of ruled surfaces, therefore it describes transverse excitations of the string and, according to [11], radial motions of the mass.

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